B.Sc Part II (Hons), Paper-3

Subring & Ideals.

Definition:_ Let $(R,+,\cdot)$ be a ring and let S be a subset of R. Then $(S,+,\cdot)$ is a subring of $(R,+,\cdot)$ if and only if $(S,+,\cdot)$ is a ring.

or

A non-empty subset S of a ring R is called a subring if (i) S is a subgroup of the additive group R and (ii) $a,b \in S$ implies $ab \in S$.

Example:_ The set S of the numbers of the form $a+b\sqrt{2}$ where a and b are integers is a commutative ring.

The set $T = \{a+0\sqrt{2}\}$ where a is an integer is a subset of S and is a commutative ring. Hence T is a subring of S.

Theorem (A):_ The intersection of two subrings is a subring.

Proof:_ Let S_1 and S_2 be two subrings of a ring R.

$$\text{let } S = S_1 \cap S_2$$

Since $0 \in S_1$ and $0 \in S_2$ $\therefore$ $0 \in S_1 \cap S_2$ i.e. $0 \in S$

$$\text{let } a,b \in S = S_1 \cap S_2.$$

Now, $a \in S_1 \cap S_2 \Rightarrow a \in S_1$ and $a \in S_2$

$b \in S_1 \cap S_2 \Rightarrow b \in S_1$ and $b \in S_2$

Since S_1 and S_2 are both subrings, we have

$a \in S_1, b \in S_1 \Rightarrow a-b \in S_1$ and $ab \in S_1$

and $a \in S_2, b \in S_2 \Rightarrow a-b \in S_2$ and $ab \in S_2$

Now $a-b \in S_1$, $a-b \in S_2 \Rightarrow a-b \in S_1 \cap S_2$

i.e. $a-b \in S$

and $ab \in S_1$, $ab \in S_2 \Rightarrow ab \in S_1 \cap S_2$

i.e. $ab \in S$

Thus $a \in S, b \in S \Rightarrow a-b \in S$ and $ab \in S$.

Hence S is a subring of R i.e. $S_1 \cap S_2$ is a subring of R.

Theorem (B):- The intersection of any family of subrings of a ring R is a subring of R.

Proof:- let S_a be a family of subrings of R. let $S = \bigcap_a S_a$

Since $0 \in S_a$ for each a $\therefore 0 \in S$

let $a, b \in S$. Then $a, b \in S_a$ for each a

Since S_a is subring.

$\therefore a-b \in S_a$ for each $a \Rightarrow a-b \in \bigcap S_a$

Hence $a-b \in S$

Also $ab \in S_a$. $\Rightarrow ab \in \bigcap S_a$. Hence $ab \in S$

Thus S is a subring of $\underline{R}$.

Ideals - def:- let $(R, +, \cdot)$ be a ring. Then $(S, +, \cdot)$ is an ideal of $(R, +, \cdot)$ if and only if.

(i) $a, b \in S \Rightarrow a + (-b) \in S$

(ii) $s \in S$ and $r \in R \Rightarrow rs \in S$ and $sr \in S$ for all $r \in R$ and for all $s \in S$.

Theo (A):- let $(R, +, \cdot)$ be a ring and $(S, +, \cdot)$ be an ideal in $(R, +, \cdot)$. Then $(S, +, \cdot)$ is a subring of R. That is every ideal is a subring.

Proof:- Given that $(S, +, \cdot)$ is an ideal in $(R, +, \cdot)$, that is it is given that (i) $a, b \in S \Rightarrow a + (-b) \in S$

(ii) $a \in S, r \in R \Rightarrow ar \in S$ and $ra \in S$. It is prove that $(S, +, \cdot)$ is a subring of R. that is it is to prove that (α) $a, b \in S \Rightarrow a + (-b) \in S$ and (β) $a, b \in S \Rightarrow ab \in S$. It is obvious that ($\alpha$) is the same as from (i). What we have to prove in fact is (β). we do this as follows let $a \in S$, $b \in S$, then $b \in R$ since S is a subset of R. Hence by (ii) $ab \in S$. Thus (β) is proved and hence the theorem.

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